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Real Analysis.

Q.1. State and prove Cauchy's general principle of convergence.

Soln Statement: → The necessary and sufficient condition for the convergence of any sequence $\{u_n\}$ is that corresponding to every arbitrary ϵ there exists a positive integer m such that

$$|u_{n+p} - u_n| < \epsilon$$

for all values of $n \geq m$ and for all positive integral values of p.

Proof: The condition is necessary:–

Let $u_n \to l$ as $n \to \infty$.

Then $|u_n - l| < \frac{\epsilon}{2}$, when $n \geq m$

Since, $n+p > m$, therefore, $|u_{n+p} - l| < \frac{\epsilon}{2}$, where p is any positive integer.

$$\therefore |u_{n+p} - u_n| = |(u_{n+p} - l) - (u_n - l)|$$
$$\leq |u_{n+p} - l| + |u_n - l|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

When $n \geq m$ and p is any positive integer.

The condition is sufficient:–

Let $|u_{n+p} - u_n| < \epsilon$, when $n \geq m$ and p is any +ve integer

Then, $u_n - \epsilon < u_{n+p} < u_n + \epsilon$, when $n \geq m$ and p is any +ve integer.

Thus, $\{u_{n+p}\}$ is bounded when $p \to \infty$.

Let N and M be the lower and upper

bounds respectively, then

$$N \geq V_n - \epsilon \text{ and } M \leq u_n + \epsilon$$

$\therefore M - N \leq u_n + \epsilon - u_n + \epsilon = 2\epsilon.$

Since, ϵ is an arbitrary small positive number, therefore, $M-N=0$ i.e. $M=N$.

$$\therefore M - \epsilon < u_{n+p} < M + \epsilon$$

i.e. $|u_{n+p} - M| < \epsilon$.

i.e. $u_{n+p} \to M$ as $p \to \infty$.

Hence, $\{u_n\}$ converges to M.

Q. 2. State and prove Cauchy's first theorem on limits.

Soln: Statement: → If the sequence $\{u_n\}$ converges to the limit l, then

$$\lim_{n\to\infty} \left(\frac{a_1 + a_2 + a_3 + \cdots + a_n}{n} \right) = l$$

Proof: Let $b_n = a_n - l$.

$$\therefore \lim_{n\to\infty} b_n = \lim_{n\to\infty} (a_n - l) = \lim_{n\to\infty} a_n - l.$$

By question, $\lim_{n\to\infty} a_n = l$

$$\therefore \lim_{n\to\infty} b_n = l - l = 0$$

$\because b_n = a_n - l$ or $a_n = b_n + l$

$\therefore a_1 = b_1 + l,\ a_2 = b_2 + l, \ldots\ldots$

Now, $\dfrac{a_1 + a_2 + a_3 + \cdots + a_n}{n} = \dfrac{(b_1+l) + (b_2+l) + (b_3+l) + \cdots (b_n+l)}{n}$

$$= \frac{nl + b_1 + b_2 + b_3 + \cdots + b_n}{n}$$

$$= l + \frac{b_1 + b_2 + b_3 + \cdots + b_n}{n}$$

or, $\lim_{n\to\infty} \dfrac{a_1 + a_2 + a_3 + \cdots + a_n}{n} = l + \lim_{n\to\infty} \dfrac{b_1 + b_2 + \cdots + b_n}{n}$

If we prove that $\lim_{n\to\infty} \dfrac{b_1 + b_2 + b_3 + \cdots + b_n}{n} = 0$, then our purpose is served.

Now, Since the sequence $\{b_n\}$ is convergent, therefore, it must be bounded. Then there exists a number K such that

$$|b_n| < K, \text{ for all } n$$

Since, the sequence $\{b_n\}$ converges to zero, therefore given $\epsilon > 0$, however small, there exists a +ve m such that

$|b_n| < \frac{\epsilon}{2}$, when $n \geq m$.

$\therefore \left|\frac{b_1+b_2+b_3+\cdots+b_n}{n}\right| = \left|\frac{b_1+b_2+b_3+\cdots+b_m}{n} + \frac{b_{m+1}+b_{m+2}+\cdots+b_n}{n}\right|$

or, $\left|\frac{b_1+b_2+b_3+\cdots+b_n}{n}\right| \leq \frac{|b_1|+|b_2|+|b_3|+\cdots+|b_m|}{n} + \frac{|b_{m+1}|+|b_{m+2}|+\cdots+|b_n|}{n}$

or, $\left|\frac{b_1+b_2+b_3+\cdots+b_n}{n}\right| \leq \frac{mK}{n} + \frac{(n-m)\epsilon}{2n}$, when $n \geq m$

or, $\left|\frac{b_1+b_2+b_3+\cdots+b_n}{n}\right| < \frac{mK}{n} + \frac{\epsilon}{2}$

Let m_1 be a positive greater than $\frac{2mK}{\epsilon}$

$\therefore$ when $n > m_1$, $n > \frac{2mK}{\epsilon}$ or, $\frac{mK}{n} < \frac{\epsilon}{2}$

Let n be the maximum of (m, m_1)

$\therefore \left|\frac{b_1+b_2+b_3+\cdots+b_n}{n}\right| < \epsilon$

$\therefore \lim_{n\to\infty} \frac{b_1+b_2+b_3+\cdots+b_n}{n} = 0$

Hence, $\lim_{n\to\infty} \frac{a_1+a_2+a_3+\cdots+a_n}{n} = l$.

Q. 3: Give an example of a monotone increasing sequence which is convergent.

Soln: Let us consider the sequence $\{u_n\}$, where, $u_n = \frac{n}{n+1}$

$\therefore u_1 = \frac{1}{2}, u_2 = \frac{2}{3}, u_3 = \frac{3}{4}, \ldots$

Evidently $\frac{1}{2} < \frac{2}{3} < \frac{3}{4} < \cdots$

That is, $u_1 < u_2 < u_3 < u_4 < \cdots$

Hence, the sequence $\{u_n\}$ is monotone increasing.

Now, $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$, which is finite.

Hence, the sequence $\{u_n\}$ under consideration is convergent.

Q.4. Discuss the convergence of the sequence $\{a_n\}$ defined by :-

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}.$$

Soln: Here, we see that

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n-2} + \frac{1}{n+n-1} + \frac{1}{n+n}$$

$$\therefore a_{n+1} = \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{(n+1)(n+1)-2} + \frac{1}{(n+1)+(n+1)-1} + \frac{1}{(n+1)+(n+1)}$$

$$\therefore a_{n+1} - a_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}$$

$$= \frac{1}{2n+1} + \frac{1}{(n+1)}\left(\frac{1}{2} - \frac{1}{1}\right)$$

$$= \frac{2n+2+2n+1-4n-2}{(2n+1)(2n+2)}$$

$$= \frac{1}{(2n+1)(2n+1)} = 0, \text{ for all } n.$$

, for all n.

or, $a_{n+1} = a_n$; for all n.

Hence, the sequence $\{a_n\}$ is monotone increasing.

Again, $a_n < \frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}$

or, $a_n < n \cdot \frac{1}{n} = 1$, for all n.

Hence, the sequence $\{a_n\}$ is bounded above and monotone increasing and therefore it must convergence.